FINDING THE MINIMUM VARIANCE PORTFOLIO USING EXCEL'S© ARRAY FUNCTIONS

By Richard R. Conn CMA, MBA, CPA, ABV, CFFA

At the heart of modern portfolio theory is the benefit derived from diversification. Simply stated, the risk of an effectively composed portfolio of securities can be significantly less than just the weighted average sum of the individual securities making up that portfolio. And here, when we speak of risk, we are referring to the variance of the returns of those individual securities. Taken to the extreme – and applied to the portfolio of ALL securities - this is the concept behind the Markowitz Bullet and the Efficient Frontier.

So, how do we know when a portfolio has been 'effectively composed'? And, what significance does technique have on the practice of business valuation? An effectively composed portfolio would be one where the relative weighting of all the constituent securities leads to the lowest possible risk – the minimum variance attainable. No other combination of those individual securities would lead to a lower expected variance. Often times in business valuation it is more desirable to a compilation of securities betas from a number of proxy firms (if they can be found) rather than just one individual firm beta. This is owing to the fact that there is a plethora of ex post beta measurement difficulties for a single individual firm that are greatly mitigated when a composite beta of a number of proxy firms can be found. If, however, a composite beta is to be employed, the question arises: 'What is the most theoretically defensible weighting of the individual proxies in order to arrive at the composite beta?' Here a very strong case can be made for using the minimum variance weighting as this mixture of securities is the only one that will lead to the lowest possible risk-to-reward ratio[1].

[1] Recognize that the ratio of expected risk-to-return is another way to define the optimal portfolio mix. That is, a portfolio that results is a variance of 0.01 and an expected return of 0.1 is superior to one of a 0.02 variance and 0.14 return. This is because the risk-to-reward ratio of the first portfolio is 0.01/0.1 = 0.1 whereas the second is 0.02/0.14 = 0.1429

The minimum variance portfolio of any collection of securities can be obtained via partial differentials in calculus, of course. But it is an arduous process – even when there are only four of five constituent securities. A much less painful process is to employ matrix mathematics and specifically the use of the ARRAY functions in Microsoft Excel©. This paper will be dedicated towards describing that technical process in as practical a manner possible. A real world solution will be provided to show the overall usefulness of this methodology and a pragmatic 'proof' of the findings will also be given. Some familiarity with basic matrix mathematics will be required.

The variance of a one-security portfolio is the sum of that security's covariance with itself – which is equal to the variance of that security. That is, $COV_{aa} = \sigma_a^2$.

The variance of a two-security portfolio is the sum of all four covariances, namely: the covariance of security a with itself, or COV_{aa} (which is just the variance of security a); the covariance of security a with security b, or COV_{ab}; the covariance of security b with security a, or COV_{ba}; and finally the covariance with security b with itself, or COV_{bb} (which is just the variance of security b). Moreover, each of these four covariances needs to be multiplied by the relative weighting as to how much of the total portfolio is invested in each security: ω_a and ω_b ;

Where

ω_a represents the proportion of the investment in security a

ω_b represents the proportion of the investment in security b,

and, $\omega_a + \omega_b = 1.0$

This arrangement can be most easily visualized in a Matrix:

MATRIX FOR THE VARIANCE OF A TWO SECURITY PORTFOLIO:

	Security a	Security b
Security a	$\omega_a\omega_a COV_{aa}$	$\omega_b\omega_a COV_{ba}$
Security b	$\omega_a\omega_b COV_{ab}$	$\omega_b\omega_b COV_{bb}$

EXAMPLE 1:

Using the matrix above as a guide, and assuming the following variables:

$COV_{aa} = 0.01$
$COV_{ab} = COV_{ba} = 0.012$
$COV_{bb} = 0.04$
$\omega_a = \omega_b = 50\%$

Therefore, the variance of this simple two-security portfolio can easily be determined just by doing the simple arithmetic of adding up the four boxes in the above matrix:

$$\sigma_p^2 = \omega_a\omega_a COV_{aa} + \omega_b\omega_a COV_{ba} + \omega_a\omega_b COV_{ab} + \omega_b\omega_b COV_{bb}$$

$$= (50\%)(50\%)(0.01) + (50\%)(50\%)(0.012) + (50\%)(50\%)(0.012) + (50\%)(50\%)(0.04)$$

$$= 0.0185$$

However, this has not taken us any closer to determining what the minimum variance portfolio would be for these two securities. In fact, just by guessing we can probably estimate a weighting that would give us a lower variance than 0.0185. For example, if we arbitrarily chose of 60/40 a/b weighting:

$$\sigma_p^2 = (60\%)(60\%)(0.01) + (40\%)(60\%)(0.012) + (60\%)(40\%)(0.012) + (40\%)(40\%)(0.04)$$

$$= 0.01576$$

We have managed to come up with a lower overall portfolio variance just by sheer chance – but still have no insight as to what the absolute minimum variance might be possible with a blend of these two securities. The nomenclature will be useful to us, however.

In a similar fashion, we can see that the variance of a three-security portfolio would be:

MATRIX abc

	Security a	Security b	Security c
Security a	$\omega_a\omega_a COV_{aa}$	$\omega_b\omega_a COV_{ba}$	$\omega_c\omega_a COV_{ca}$
Security b	$\omega_a\omega_b COV_{ab}$	$\omega_b\omega_b COV_{bb}$	$\omega_c\omega_b COV_{cb}$
Security c	$\omega_a\omega_c COV_{ac}$	$\omega_b\omega_c COV_{bc}$	$\omega_c\omega_c COV_{cc}$

Where

$$\omega_a + \omega_b + \omega_c = 1.0$$

And a five-security portfolio, which will be the basis for our real-world data, is as follows:

MATRIX abcde

	Security a	Security b	Security c	Security d	Security e
Security a	$\omega_a\omega_a COV_{aa}$	$\omega_b\omega_a COV_{ba}$	$\omega_c\omega_a COV_{ca}$	$\omega_d\omega_a COV_{da}$	$\omega_e\omega_a COV_{ea}$
Security b	$\omega_a\omega_b COV_{ab}$	$\omega_b\omega_b COV_{bb}$	$\omega_c\omega_b COV_{cb}$	$\omega_d\omega_b COV_{db}$	$\omega_e\omega_b COV_{eb}$
Security c	$\omega_a\omega_c COV_{ac}$	$\omega_b\omega_c COV_{bc}$	$\omega_c\omega_c COV_{cc}$	$\omega_d\omega_c COV_{dc}$	$\omega_e\omega_c COV_{ec}$
Security d	$\omega_a\omega_d COV_{ad}$	$\omega_b\omega_d COV_{bd}$	$\omega_c\omega_d COV_{cd}$	$\omega_d\omega_d COV_{dd}$	$\omega_e\omega_d COV_{ed}$
Security e	$\omega_a\omega_e COV_{ae}$	$\omega_b\omega_e COV_{be}$	$\omega_c\omega_e COV_{ce}$	$\omega_d\omega_e COV_{de}$	$\omega_e\omega_e COV_{ee}$

Where

$$\omega_a + \omega_b + \omega_c + \omega_d + \omega_e = 1.0$$

In the same manner as was demonstrated with the two-security example, the variance for a five-security portfolio could be determined by adding the results of the above 25 cell matrix. And the minimum variance portfolio could be determined by partial derivatives – but the process would be very onerous. Note that without the ω elements, the above matrix becomes:

MATRIX COV_{ij}

	Security a	Security b	Security c	Security d	Security e
Security a	COV_{aa}	COV_{ba}	COV_{ca}	COV_{da}	COV_{ea}
Security b	COV_{ab}	COV_{bb}	COV_{cb}	COV_{db}	COV_{eb}
Security c	COV_{ac}	COV_{bc}	COV_{cc}	COV_{dc}	COV_{ec}
Security d	COV_{ad}	COV_{bd}	COV_{cd}	COV_{dd}	COV_{ed}
Security e	COV_{ae}	COV_{be}	COV_{ce}	COV_{de}	COV_{ee}

… and that all 25 of these covariances can be determined from historical public securities data. In fact, the covariance of 5 actual mid-cap oil and gas producers is as follows[2]:

	MATRIX C				
Symbol	BNP	KEY	NKO	PGF	VET
BNP	0.0072461	0.0027012	0.0031799	0.0052070	0.0039898
KEY	0.0027012	0.0051091	-0.0002237	0.0032930	0.0021650
NKO	0.0031799	-0.0002237	0.0128479	0.0038362	0.0033431
PGF	0.0052070	0.0032930	0.0038362	0.0084014	0.0038015
VET	0.0039898	0.0021650	0.0033431	0.0038015	0.0055136

We shall call the above 'Matrix C'. What we require, however, is Matrix C^{-1}. Recall that the C^{-1} syntax for a matrix indicates that inverse of Matrix C (see Appendix A for a refresher on the inversion of Matrices). These can be extremely tedious to calculate manually. Fortunately, the Excel© MINVERSE function, easily produces the inversion of any matrix. Therefore, Matrix C^{-1} via the MINVERSE function is:

	MAXTRIX C^{-1}					
Symbol	BNP	KEY	NKO	PGF	VET	
BNP	305.9915	-36.1442	-10.2225	-116.053	-121.013	
KEY	-36.1442	306.4613	59.62163	-95.8814	-64.2245	
NKO	-10.2225	59.62163	107.6456	-42.7396	-51.8152	
PGF	-116.053	-95.8814	-42.7396	263.4952	-34.1337	
VET	-121.013	-64.2245	-51.8152	-34.1337	349.1066	
TOTAL	22.5587	169.8329	62.48997	-25.3125	77.92021	307.4893

[2] All five firms trade on the Toronto Stock Exchange and the covariance's have been determined from the 60 monthly returns ending in June 2011.

The total for each column, and the total of all the column totals have been included here for convenience (we will see why these are important in a second) – but they are not technically part of the inverse matrix. Once having attained matrix C^{-1} it is reasonably easy to show that the minimum variance portfolio can be calculated as[3]:

$$\omega = \upsilon C^{-1} / \upsilon C^{-1} \upsilon^{T}$$

Where

ω is the minimum variance weighting, and, in our specific example will be comprised of ω_1, ω_2, ω_3, ω_4 and ω_5

υ is the single row matrix of five 1's as in $\upsilon = \{1, 1, 1, 1, 1\}$

C^{-1} is the inverse of matrix C

υ^{T} is the transposition of matrix υ, therefore it is a single column matrix of five 1's

The product of the two matrices υC^{-1} can be easily determined by using Excel's© MMULT function, which multiplies the two named arrays. This gives:

BNP	KEY	NKO	PGF	VET
22.5587	169.8329	62.48997	-25.3125	77.92021

Notice that this is just the column totals of matrix C^{-1} above.

And, in order to obtain the denominator amount of $\upsilon C^{-1} \upsilon^{T}$ we simply multiply the above 5-cell matrix υC^{-1} by the matrix υ^{T} (using Excel's© MMULT function) and we obtain the single-cell matrix results of 307.4893. Note that this result is simply the sum total of the column totals in matrix C^{-1}.

So while the matrix nomenclature of

$$\omega = \upsilon C^{-1} / \upsilon C^{-1} \upsilon^{T}$$

[3] See, for example, Marek Capinksi & Tomasz Zastawniak, *Mathematics for Finance: An Introduction to Financial Engineering*, Springer-Verlag London Ltd., pg. 109

may not be intuitive to those of us not regularly accustomed to working with matrices, it is simply telling us that the minimum variance weighting for each security can be found by dividing the column total with the table total as in:

ω_1 = 22.5587 / 307.4893 = 7.336418%

ω_2 = 169.8329 / 307.4893 = 55.232133%

ω_3 = 62.48997 / 307.4893 = 20.322647%

ω_4 = -25.3125 / 307.4893 = -8.231994%

ω_5 = 77.92021 / 307.4893 = 25.340787%

Note that the sum of $\omega_1+\omega_2+\omega_3+\omega_4+\omega_5$ does total to 100% as we would expect it to and that, by the nature of ω_4 weighting (for the PGF security) being negative we are being instructed to sell-short this stock in order to achieve the lowest possible portfolio variance.

Now, having obtained what is purported to be (but not yet proven) the unique weighting arrangement that should lead us to the minimum portfolio variance, we can now combined the investment weights as determined above with the Covariances as presented in matrix C as represented in **Matrix abcde** and obtain:

MATRIX $\omega C\omega^T$

	MAXTRIX $\omega C\omega^T$					
	BNP	KEY	NKO	PGF	VET	
BNP	3.9001E-05	0.000109453	4.74102E-05	-3.14467E-05	7.41737E-05	
KEY	0.00010945	0.001558589	-2.51097E-05	-0.000149723	0.000303021	
NKO	4.741E-05	-2.51097E-05	0.000530633	-6.41776E-05	0.000172167	
PGF	-3.145E-05	-0.000149723	-6.41776E-05	5.69332E-05	-7.93025E-05	
VET	7.4174E-05	0.000303021	0.000172167	-7.93025E-05	0.000354061	
Minium Portfolio Variance	0.00023859	0.00179623	0.000660922	-0.000267717	0.000824119	0.00325215
Minium Monthly Portfolio STDEV						5.7%
Minimum Annualized Portfolio STDEV						19.75%

This leads us to a minimum portfolio variance of 0.00325215 (simply the sum of the 25 cells in the matrix). Stated as a standard deviation, this equates to 5.7%. And, finally, because monthly variances had been used in the original 60 month covariance samples, in order to restate this monthly volatility in terms of an annualized standard deviation, we have multiplied the 0.057 by $\sqrt{12}$ to get 19.75% volatility per year.

THE BENEFITS OF DIVERSIFICATION

Before we consider the validity of this result, and perhaps consider if there is any logical proof that could be developed to show that 0.00325215 monthly variance is, indeed, the absolute minimum that could possibly be obtained with these five specific securities, there is another more important question we should turn our attention to. Is there a benefit of diversification here? We can determine this by multiplying each security's stand-alone monthly variance by the prescribed weightings (the ω's determined as above) and then comparing this overall weighted average variance to the 0.00325215. For example:

TABLE OF WEIGHTED AVG. STAND-ALONE VARIANCES (i.e. outcome if there were no benefits to diversification)

Symbol	ω	Stand-alone σ^2	$\omega\sigma^2$
BNP	7.3%	0.007246	0.0005316
KEY	55.2%	0.005109	0.00282189
NKO	20.3%	0.012848	0.00261104
PGF	-8.2%	0.008401	-0.0006916
VET	25.3%	0.005514	0.0013972
	100.0%		
		W.Avg Monthly Variance	0.00667012
		W. Avg Monthly Standard Dev.	8.17%
		W. Avg Annualized Standard Dev.	28.29%

From the table above we can see that there are considerable benefits from diversification. That is, the overall portfolio variance, if it were the simple weighted average sum of all the individual securities variances would have amounted to 0.00667 rather than the much reduced diversified finding of 0.00325. On an annualized volatility basis this equates to a 28.3% compared with the expected 19.7%. A substantial reduction in risk occurs as a result of diversification. Note that, had each security been perfectly correlated with each other (i.e. they all reported a correlation coefficient of 1.0), then there would have been no benefits of diversification, and the 28.3% annual volatility would have resulted from the proposed weighting scheme.

MINIMUM PORTFOLIO VARIANCE QUASI-PROOF

How do we know that the proposed weightings of

Symbol	ω
BNP	7.3%
KEY	55.2%
NKO	20.3%
PGF	-8.2%
VET	25.3%
	100.0%

leads to the to the lowest possible attainable portfolio variance? As previously mentioned, a partial derivative solution of $d\sigma_p^2$ with respect to $d\omega$ could be developed. However, because there would be 25 individual ω_i variables in this equation, such a calculation would be extremely onerous. While it could certainly not be considered an absolute 'proof' an easier approximation would be to employ Monte Carlo testing of the proposed ω's. For example, if we arbitrarily devised five ω's (i.e. ω_1 through ω_5) that totaled to 1.0 and substituted these into MATRIX $\omega C \omega^T$ above, we could then compare the results with the 0.00325215 alleged minimum. If our arbitrarily selected ω's resulted in a higher σ_p^2 that may give us some comfort that the 0.003251215 solution was the ultimate minimum – but a sample of one observation would never be statistically reliable and would not provide us with very much confidence.

On the other hand, if we repeated the same arbitrary selection process whereby the ω's were being determined entirely by an unbiased random number generator and repeated this test 10,000, or 50,000 or 100,000 times – then the results of these tests may be very convincing if we were to find that none of those 50,000 outcomes (for example) produced a σ_p^2 lower than 0.00325215. Just such an experiment was constructed for this case (using the Excel© RANDBETWEEN function).

In order to determine the randomly generated ω's wherein $\sum \omega_i$ in each instance does sum to 1.0, use the following:

$$\omega_i = \text{RANDBETWEEN}(-1000,1000) / (\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5)$$

In other words, for each of the five random ω's an integer of between minus 1,000 to plus 1,000 inclusive can result and since all five of these random outcomes is being divided by the sum of the five, the proportionate amount of each will always total 100%[4]. These five random proportionate outcomes are then input into Matrix $\omega C \omega^T$ from which the resultant σ_p^2 would be recorded. Some simple VBA code was written to repeat this random process 50,000 times and record all the resultant outcomes.

Of the 50,000 outcomes, only the following 8 had resultant annualized standard deviations less than 20.0%.

[4] Note that it is mathematically possible, under this construct, that the net result of the sum of the five ω's amounts to a negative total. This would indicate a net short position of the portfolio and is a viable solution (although it would be unlikely that this would ever lead to a minimum variance).

	ω_1	ω_2	ω_3	ω_4	ω_5	$\sum\omega_i$	
Trial #	**BNP**	**KEY**	**NKO**	**PGF**	**VET**		**Port Stdev**
5024	0.176132	0.475198	0.176132	-0.05176	0.224299	1.000	19.94%
7334	-0.00123	0.5187	0.193746	0.058246	0.230533	1.000	19.99%
9101	0.027647	0.474713	0.20499	-0.08564	0.378287	1.000	19.95%
14463	0.128702	0.509424	0.173398	-0.04685	0.235326	1.000	19.84%
18475	0.131201	0.535248	0.241514	-0.06136	0.153394	1.000	19.90%
27794	0.033999	0.48462	0.168915	0.004317	0.308149	1.000	19.89%
34591	0.055349	0.442789	0.211283	0.00479	0.28579	1.000	19.98%
36612	0.148568	0.45705	0.187466	-0.06267	0.269584	1.000	19.91%

Where 19.75% is the proposed absolute minimum annualized standard deviation, the closest simulated outcome to that is 19.84% of trial #14,463. Such robust results provide a good deal of confidence in the derived minimum variance weighting solution. Less confidence would have been had if, for example, 1,000 of the random outcomes had generated results whereby $\sigma_p < 0.20$. And, because of the easily repeatable design of the model, one can simply continue running the 50,000 random sample observation to gain an even greater confidence in the results. If, after running the experiment 20 times (i.e. 1 million random outcomes) there is still not a single occurrence of a annualized standard deviation less than 0.1975, then one should be able to profess a high degree of confidence that the minimum variance methodology as outlined above does, indeed, provide accurate results.

IMPACT UPON THE WEIGHTING OF A GROUP BETA

Now that we have what we believe to be the 'optimal' weightings for this five-security portfolio, we can examine how our results will change the group beta. Without this information, two common approaches to arriving at a group weighted average beta would have simply to take the equally weighted average (i.e. 20% of each stand alone beta); or, to use each firm's current market capitalization as the appropriate weighting. Those approaches would have resulted as:

TWO COMMON APPROACHES TO WEIGHT GROUP BETAS					
Symbol	Market Cap as at mid 2011 ($B)	Market Cap Weighting	Market Beta as regressed w TSX Index	Market Cap Weighted Beta	Equally Weighted Beta
BNP	3.757	20.4%	1.13	0.2304	0.226
KEY	3.079	16.7%	0.63	0.1053	0.126
NKO	3.378	18.3%	1.19	0.2181	0.238
PGF	3.931	21.3%	1.15	0.2453	0.23
VET	4.282	23.2%	1.06	0.2463	0.212
	18.427	100.0%		**1.0454**	**1.032**

In this instance either the market-cap weighting or the equal weighting would have produced indistinguishable results. The overall beta for this group, would have been 1.0

This is not true, however, of the minimum variance weigting – nor would we expect it to be. We have already seen that there are considerable benefits from diversification in reducing volatility by selecting the lowest risk portfolio in this group. Therefore, we should expect a corresponding reduction in group beta when the minimum variance beta is applied. And this is, if fact, the case:

Symbol	ω	Market Beta as regressed w TSX Index	Minimum Variance Weighted Beta
BNP	7.336%	1.13	0.083
KEY	55.2%	0.63	0.348
NKO	20.3%	1.19	0.242
PGF	-8.2%	1.15	-0.095
VET	25.3%	1.06	0.269
	100.0%		0.847

Under this scenario, only a 0.85 group beta would be applied.

CONCLUSIONS

The minimum portfolio variance for any given selection of securities can easily be determined by the $\omega = \upsilon C^{-1} / \upsilon C^{-1} \upsilon^{T}$ formula and Microsoft Excel's© ARRAY functions make calculating the optimal weightings (the ω's) relatively quick and painless.

APPENDIX A – MATRIX REFRESHER

IDENTITY MATRIX:

An Identity Matrix is one of n rows and n columns where the main diagonal cells (beginning in the upper left and descending to the lower right) is equal to 1 and all the other cells = 0. A 5 x 5 identity matrix is:

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

INVERSE MATRIX:

An Inverse Matrix is one whereby multiplying the original matrix by the inverse matrix will result in an identity matrix. The simplest of all possible examples is a single cell matrix of a value 5.0. The inverse matrix (obviously also a single cell matrix) must be 5^{-1} or 1/5 as this is the only possible value that would, when multiplied by the original 5.0 give the identity matrix value of 1.0

Using Excel's© MMULT function it can be shown that multiplying

	MAXTRIX C				
Symbol	BNP	KEY	NKO	PGF	VET
BNP	0.0072461	0.0027012	0.0031799	0.0052070	0.0039898
KEY	0.0027012	0.0051091	-0.0002237	0.0032930	0.0021650
NKO	0.0031799	-0.0002237	0.0128479	0.0038362	0.0033431
PGF	0.0052070	0.0032930	0.0038362	0.0084014	0.0038015
VET	0.0039898	0.0021650	0.0033431	0.0038015	0.0055136

By

	MAXTRIX C^{-1}				
Symbol	BNP	KEY	NKO	PGF	VET
BNP	305.9915	-36.1442	-10.2225	-116.053	-121.013
KEY	-36.1442	306.4613	59.62163	-95.8814	-64.2245
NKO	-10.2225	59.62163	107.6456	-42.7396	-51.8152
PGF	-116.053	-95.8814	-42.7396	263.4952	-34.1337
VET	-121.013	-64.2245	-51.8152	-34.1337	349.1066

Does produce the identity matrix of:

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

MULTIPLYING MATRICES:

In order to find the product of two matrix, it is first necessary that, where m x n is the number of "m" rows and "n" columns of the first matrix, and o x p is the number of "o" rows and "p" columns of the second matrix, that n = o. That is, the first matrix must have the same number of columns as the second matrix has rows.

Example 1) A square 3 x 3 Array times B square 3 x 3 Array

Array A			Array B				Product A x B		
a	b	c	j	k	l	=	aj + bm + cp	ak + bn + cq	al + bo + cr
d	e	f	m	n	o	=	dj + em + fp	dk + en + fq	dl + eo + fr
g	h	i	p	q	r	=	gj + hm + ip	gk + hn + iq	gl + ho + ir