CONVERTIBLE BONDS: Black Scholes vs. Binomial Models

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For years I have believed that the Black-Scholes (B-S) option pricing model should not be used in the valuation of convertible debt. But recent events have conspired to make me question how well-founded this belief is in actual financial theory. For those of us that have been engaged in the pricing of convertible securities in the past, hopefully this article will serve as a useful refresher – for those that haven't it should be a good introductory overview of the process.

Upon reflection, my reluctance to engage B-S in this application is probably based more upon the observation that the professional valuation community tends to over-use B-S. For those of us without extensive backgrounds in math (regrettably, myself included), we like using B-S because it offers us a closed-form solution to plain vanilla put & call options that does not require the use of calculus[1]. Indeed, it is easy to construct a perfectly accurate European B-S model in Excel™. Further, anyone reasonably familiar with using a standard normal distribution table from the back of any college statistics textbook could arrive at a Black-Scholes solution just with pencil and paper (but you wouldn't want to have to do it twice). Black-Scholes has become widely accepted for use in stock-based-compensation financial reporting and is frequently referred to in the financial press. All of the public stock exchanges use B-S to report the 'implied volatility' statistics published daily with all the traded option prices. As a result, whenever we come across a valuation that involves a derivative, many tend to force the application of B-S into the problem regardless of whether it is the most appropriate methodology for the circumstances.

To be clear, convertible bonds (CB) can be very complex securities. And, for those that include periodic call (redeemable) and/or put (retractable) features, or floating coupon rates, or are exchangeable into other exotic securities, or the underlying has an irregular dividend structure – then B-S can never be relied upon to produce a reasonable approximation of the CB's current FMV. For these more complicated indentures, a path-dependent lattice model (perhaps the Binomial) must be employed. Or really sophisticated practitioners might apply a finite difference method - and engage a math PhD to explain how it works. But that does not mean, as it turns out, that the humble B-S call option formula does not serve a useful purpose in the valuation of convertible securities.

To simplify the exposition, we will, initially speak of only the most basic, plain vanilla convertible bond possible: no put/call features (i.e. neither the issuer is permitted to redeem the bond early nor can the bondholder retract the bond prior to maturity); there currently are no dividends paid on the underlying common stock nor are there any expected prior to bond maturity; early conversion at the investor's option is possible (i.e. it is an American option) – but this point is moot as it is almost never advantageous for a

[1] To be precise, the B-S formula does require the integration of the standard normal distribution, but Excel™ provides a reliable approximation for this function via 'NORMSDIST'.

CB holder to convert prior to maturity when the underlying common stock does not pay dividends; forced conversion is not possible (i.e. the issuer cannot instigate a conversion); the semiannual coupon payments are based upon a fixed rate of interest. Further, we are going to presume that the issuer is a private company, but strong public guideline firms have been found to provide proxy values for the market rate of interest on a comparable debt-only instrument, as well as a proxy for the firm's annual stock price volatility. A significant private placement was recently completed for our subject firm at \$100/share. And, finally, the dilutive effects of conversion are not expected to be significant.

$CB(S,X,r_f,r_d,r_c,T,\sigma,CR,FV)$. That is, the current FMV of the CB is a function of nine inputs – but we are going to deem all but two of those constant (sort of). They are:

- S ≡ Spot Price of Underlying Common Shares **(\$100/share** on valuation date)
- X ≡ Strike Price (varies, based upon the market value of the pure-debt)
- r_f ≡ Risk-free rate = **5.0%** continuously compounded (cc)
- r_d ≡ Market rate of interest on similar pure-debt = **9.0% cc**
- r_c ≡ Coupon Rate = **7.0% APR** (i.e. 3.5% coupon semi-annually)
- T ≡ Time to Maturity = **2 Years**
- σ ≡ Annual volatility on similar public shares = **40%**
- CR≡ Conversion Ratio = **10** shares received for each bond surrendered
- FV≡ Face Value ≡ Maturity Value of bond = **\$1,000**

CONVERTIBLE BONDS – TWO STATES OF NATURE

There are generally two states of nature for a CB and its 'value components' are contingent upon which state it is in at the time. One state is where the embedded option is far out-of-the-money. In this case, little if any value can be attributed to the 'optionality'. At this point virtually all of the current FMV of the CB will stem from the 'pure bond' (i.e. a comparable bond with no conversion features) component – so, if we can find a proxy for the pure bond rate, then the present value of the maturity amount and future coupon stream will accurately reflect the value of the CB.

The second state of nature begins when the option approaches being at-the-money and the probability of ultimate conversion starts to increase. Now the CB will begin performing more like equity and, as the embedded option becomes increasingly in-the-money, the more the value of the CB will parallel the underlying movements of the stock price[2]. The greater the probability of conversion, the more we can say the converted value will dominate over the pure bond value.

Obviously, once converted, the FMV will simply be the number of shares received multiplied by the share price. In this second state of nature, just before conversion, there will be, however, three distinct value components to the CB:

[2] So when the 'convert-to' common shares are highly liquid and publicly traded, it is relatively simple to determine what the CB value is at this state of nature. This is not the case when the shares are illiquid.

i) Strike Price Value
ii) Intrinsic Value
iii) 'Optionality' Value

We defer the discussion of the strike price value – which is the key to using Black-Scholes in CB valuations, and begin with Intrinsic Value. Intrinsic Value is simply the amount by which the current share price exceeds the strike price. If you can buy a share currently trading at $100 for $75, there is obviously $25 of intrinsic value. This is the 'in-the-money' component and a CB will always reflect this value or else a riskless arbitrage would be possible.

The 'Optionality Value'[3] is akin to future potential. If you own a plain vanilla call option with a strike of $75, a current spot price of $100 and the potential that the stock may go up to $120 prior to the expiration of the call option, then optionality is represented by the possible move from $100 to $120. The value of this optionality is directly dependent upon the probability of the share price appreciation. Note that, if you simply outright purchased a share at $100, then it too has this potential to appreciate to $120 and therefore the optionality value is imbedded in the $100 price. Indeed, this is the very reason why investors purchase shares. However, the price of optionality is different in a derivative compared with just buying the underlying security. This is because when you purchase the underlying you have actually paid for the probability of appreciation and put your money at risk. In contrast, when you hold a call with a fixed strike price that is already in-the-money and has the potential to go deeper, you have the luxury of waiting to see if, in fact, the $120 share price is attained without the necessity of putting any additional funds at risk. In this sense, optionality can be measured in the difference of exercising the option early (i.e. paying $75 for a $100 stock with $25 intrinsic value) and hoping the $120 share price is subsequently achieved, or, waiting until there is absolute certainty that the $120 price is met (at which time the intrinsic value of the option would be $45) and exercising then.

Finally we get to the Strike Price component of CB value. This is equivalent to the current FMV of the pure-bond and floats as the market value of the pure debt changes[4].
The cost of the CB strike price is the opportunity cost of relinquishing the present value of the pure bond value at the time of conversion. Optimally, the best time for a CB holder to convert would be when the differential between the share price (adjusted for conversion ratio) and bond value was at its greatest. Hypothetically this would be when the share price was ascending and the bond value decreasing – but it is difficult to imagine circumstances when a firm's market equity and bond value would be moving in opposite directions.

[3] This is my own term. Some might refer to this as the Time Value component of an option, but this might be confused with a strict time-value-of-money concept – which 'optionality' is not.
[4] In a truly European-style CB, where conversion can only take place on the maturity date, a strong case can be made for the fact that the Strike Price will be the fixed Face Value of the bond, plus any accrued coupon interest that might also be sacrificed in the conversion.

AN EXAMPLE

Using the previously defined: CB(S=$100, X=PV pure bond, r_f = 5%cc , r_d = 9%cc, r_c = 7% APR, T = 2 Years, σ = 40%,CR = 10, FV = $1,000) we are going to compare how a B-S derived CB value would compare with a simple Binomial model solution (employing the methodology suggested by John C. Hull in his authoritative text "Options, Futures and Other Derivatives".

The current FMV of the pure-bond must be equal to the present value (PV) of the Face Value plus the PV of an annuity of four coupon payments of $35 (7% APR paid semi-annually) each:

Pure Bond $= \$1{,}000e^{(-0.09 \times 2)} + (\$35.00/0.04603^*)(1 - 1.04603^{-4})$
$= \mathbf{\$960.53}$

(* note that the semi-annual effective rate of interest for 9.0%cc is 4.603% rounded)

So this provides us with the current strike price for the B-S solution:

$$\text{B-S Call} = SN(d1) - Xe^{(-rf \times T)} N(d2)$$

Where:

S, X, r_f, T are as previously defined, and;

$d1 = [\ln (S/X) + (r_f + \sigma^2/2)(T)] / \sigma\sqrt{T}$

$d2 = d1 - \sigma\sqrt{T}$

e = the natural log base

N(x) is the normal cumulative probability function (i.e. NORMSDIST)

Therefore (and, accommodating for the fact that each bond converts to 10 shares):

$$\begin{aligned}\text{B-S Call} &= \$100 \times 10 \cdot N(d1) - \$960.53e^{(-0.05 \times 2)} \cdot N(d2) \\ &= \$1{,}000 \cdot 0.702224 - \$869.12 \cdot 0.486089 \\ &= \mathbf{\$279.75}\end{aligned}$$

Note that a B-S Call premium would contain both the intrinsic value and the optionality components of value. This CB, for example, could be converted today in order to realize $100 x 10 - $960.53 = $39.47 of 'in-the-money' intrinsic value. That must mean that the remainder of the Black-Scholes premium $279.75 - $39.47 = $240.28 relates to the optionality value component which reflects the potential of the future share price to exceed the existing $100.

As previously discussed, the B-S derived FMV on this very plain vanilla CB must be the sum of the Strike Price + Intrinsic Value + Optionality. Specifically it is $960.53 + $279.75 = **$1,240.28**

How does this compare with the more elaborately derived binomial model?

B-S Call Method	$1,240.28
Binomial Model[5]	$1,240.06

Frankly, one might have expected a greater difference given that the B-S formula is continuous where as the binomial model is based upon a limited number of discrete time steps. Moreover, the B-S formula originates from a risk-neutral world where only the risk-free rate is employed. In contrast, the binomial model allows for either the risk-free or market rate of debt (or a blend of both) to be used at each discrete time step.

SENSITIVITY TESTING

In order to gain some assurance that the similarity in the two approaches was not due to just random chance, some sensitivity testing was undertaken. The convertible bond value is dependent upon nine input variables: $CB(S,X,r_f,r_d,r_c,T,\sigma,CR,FV)$. However, we will assume all the base-case inputs as above and only change one input at a time. And, for brevity, the notation will only indicate the one input that has been altered. So, $CB(r_d = 6\%cc)$, for example, indicates that only the market rate of pure-debt interest was changed to 6% whereas all the other inputs are as assumed in the above-noted base case.

Inputs	B-S CB	Binomial CB	Difference
BASE CASE	$ 1,240.28	$ 1,240.06	$ 0.22
CB(S = 96.05)	$ 1,213.06	$ 1,214.58	$ (1.52)
$CB(r_f$ = 2% cc)	$ 1,215.30	$ 1,235.33	$ (20.03)
$CB(r_d$ = 13% cc)	$ 1,202.66	$ 1,198.00	$ 4.66
$CB(r_d$ = 6% cc)	$ 1,272.88	$ 1,273.80	$ (0.92)
$CB(r_c$ = 10% APR)	$ 1,271.30	$ 1,288.44	$ (17.14)
CB(T = 4)	$ 1,336.37	$ 1,346.53	$ (10.16)
CB(σ = 20%)	$ 1,143.63	$ 1,143.71	$ (0.08)
CB(σ = 60%)	$ 1,337.02	$ 1,336.50	$ 0.52
CB(CR = 12)	$ 1,391.51	$ 1,392.82	$ (1.31)

The largest single difference between the two methodologies was in relation to a significant drop in the risk-free rate. This is because B-S only relies upon the risk-free rate and is very sensitive to it. In contrast, the binomial model uses a varying blend of both the risk-free rate and the pure debt rate and is therefore more insulated when one

[5] Actually, three independent binomial versions were tested using a different number of discrete time steps: 16, 64 and 128 steps. The output differences between the three models were not significant, but the 128 step model is presumed to be the most precise and is the version relied upon herein.

changes and the other is held constant (which is economically unrealistic, but was useful for our sensitivity analysis). On the whole, however, the differences between the two methodologies are quite insignificant. Only two of the trials resulted in an absolute difference that equated to slightly more than 1% of the combined average B-S and Binomial output of that trial.

CONCLUSIONS

Contrary to my long-standing belief, it appears the Black-Scholes call option formula does provide a reasonable approximation for the unrealistically simple convertible bond that has no put-call (retractable/redeemable) provisions on a stock that pays no dividends[6]. This is a useful observation because it provides a reasonability check on any binomial model that is under construction. If all the bells and whistles of the typical CB can be temporarily stripped away or zeroed out, then the underlying binomial tree should approximately return the same current FMV as the B-S approach. From there it is a matter of considering how the more sophisticated attributes change CB value: from the CB-holder's perspective, a call feature decreases CB value whereas a put will increase it; dividends detract from the optionality value but, if early conversion is always possible, dividend yields in excess of the coupon rate, or special dividends will generally increase CB value. Of course, the more complex the CB indenture becomes, the less likely a simple B-S approach is going to capture current FMV for reasons other than pure chance.

One thing that is absolutely certain is that the amount of time and effort required to apply the B-S methodology is only a mere fraction of the effort involved in constructing even a relatively simply binomial model from scratch. Therefore it only seems prudent that first part of any CB valuation assignment would start with a B-S solution if only to serve as a guideline for what the security value would be if all the complexities were stripped away.

[6] Black-Scholes can be easily modified to provide a reasonably reliable approximation of option value for an underlying with a constant and predictable dividend yield. For space considerations, this alternative has not been considered here, but obviously this feature will increase the usefulness of the B-S CB methodology.